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ON UNIFORM LINEAR INVARIANT RELATIONS OF THE EQUATIONS OF DYNAMICS*

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In a tangential stratification of the configuration manifold of a mechanical system, the submanifolds of its trajectories specified in local coordinates by equations that are linear and homogeneous, with respect to velocities, are discussed. The local conditions for the existence of some of such submanifolds in a structural form are established. The results obtained are illustrated by examples taken from solid dynamics.

1. Let $q \in R^n$ be the Lagrangian coordinates of a holonomic mechanical system, $T = {}^1/_2 (a(q)q^{\cdot}, q^{\cdot})$ its kinetic energy, and $F(q) \in R^n$ the generalized force. The equation of motion can be written in a form which can resolve in terms of accelerations,

$$q^{\cdot\cdot} = -(\Gamma q^{\cdot}, q^{\cdot}) + F \tag{1.1}$$

or, when the velocity field $q^{\cdot} = f(q)$ is specified.

$$(f, \nabla) f = F \tag{1.2}$$

Here $\Gamma(q)$ is the connectivity object (see /1/), ∇ denotes covariant differentiation, $(\xi, \eta) = \xi_i \eta^i$; the repeating index is understood to mean summation from 1 to n.

Definition. The relations

$$\varphi_1(q, q^{\cdot}) = 0, \ldots, \varphi_m(q, q^{\cdot}) = 0 \quad (m \leqslant 2n) \tag{1.3}$$
$$\operatorname{rank} \| \partial\varphi/\partial q, \quad \partial\varphi/\partial q^{\cdot} \| = m$$

form, in a certain domain of variation of the variables q and $q^{\cdot}$ an invariant ensemble for the system of differential equations $q^{\cdot\cdot} = G(q, q^{\cdot}) \in R^n$, if for each $\alpha = 1, \ldots, m$ the expression

$$\frac{d\varphi_\alpha}{dt} = \left(\frac{\partial \varphi_\alpha}{\partial q}, q^{\cdot}\right) + \left(\frac{\partial \varphi_\alpha}{\partial q^{\cdot}}, G\right)$$

has the form

$$\frac{d\varphi_\alpha}{dt} \equiv \sum_{\beta=1}^{m} \varkappa_{\alpha\beta}(q, q^{\cdot}) \varphi_\beta \tag{1.4}$$

($\varkappa_{\alpha\beta}$ are the continuous functions).

In the tangential destratification TM of the configuration manifold M of a mechanical system, Eqs.(1.3) define locally a certain submanifold. Under conditions (1.4), the integral curve of the equations of motion, which has a common point with this submanifold, lies on it, i.e. the given submanifold is integral.

Let us consider the question of the existence, for Eqs.(1.1), of an ensemble of invariant relations of the form

$$\langle \lambda_{n-m+1}, q^{\cdot} \rangle = 0, \ldots, \langle \lambda_n, q^{\cdot} \rangle = 0 \quad (m \leqslant n-1) \tag{1.5}$$

where the vectors $\{\lambda_\alpha(q)\}$ are linearly indepenent of each point, $\langle \xi, \eta \rangle = (a\xi, \eta)$. In doing so, we shall confine ourselves to studying two extreme cases: $m = n-1$ and $m = 1$.

Theorem 1. Let $F \neq 0$. An ensemble of the $(n-1)$–th invariant relations (1.5) exists if and only if the lines of force are geodesic lines of the Riemann manifold $(M, \langle\ ,\ \rangle)$. For $F \equiv 0$, the system has ∞^n of such invariant ensembles.

This theorem is a corollary of Theorem 3 proved below. For $n = 2$, it is given in /2/.

For $F = \operatorname{grad} U(q)$, the condition of the theorem is written analytically as

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$$\partial_1 (\Delta_1 U)/\partial_1 U = \ldots = \partial_n (\Delta_1 U)/\partial_n U \tag{1.6}$$
$$\Delta_1 U = (\text{grad } U,\ a^{-1} \text{ grad } U), \quad \partial_i = \partial/\partial q^i$$

(see /1/).

We note that in the case of $(n-2)$ first linear integrals (general) in a holonomic system Theorem 1 give the necessary condition for the existence of one more linear integral or a single invariant relation $\langle \lambda_n (q), q^{\cdot} \rangle = 0$:

$$d/dt \langle \lambda_n, q^{\cdot} \rangle = \varkappa (q, q^{\cdot}) \langle \lambda_n, q^{\cdot} \rangle \tag{1.7}$$

Example 1. A solid rotates about a fixed point in a force field

$$U = xu + yv + zw + {}^1/_2 \varepsilon (Au^2 + Bv^2 + Cw^2)$$

which is the superposition of a uniform gravitational field and the Brun field /3/. Here A, B and C are the main moments of inertia of the body for the fixed point; (x, y, z) and (u, v, w) are the coordinates on the main axes of the centres of inertia of the body and of a unit vector of the vertical respectively, and ε denotes a constant parameter.

We have

$$\Delta_1 U = a [yw - zv + \varepsilon (B - C) vw]^2 + b [zu - xw + \varepsilon (C - A) uw]^2 + c [xv - yu + \varepsilon (A - B) uv]^2$$

$(a = A^{-1}, b = B^{-1}, c = C^{-1})$. The condition (1.6) takes the form

$$\partial_1 U \partial_2 (\Delta_1 U) - \partial_2 U \partial_1 (\Delta_1 U) = 0 \tag{1.8}$$

where q^1 and q^2 are any two of the three quantities u, v, w, and the third quantity is replaced by the expression $\pm[1 - (q^1)^2 - (q^2)^2]^{1/2}$ in U and $\Delta_1 U$

Setting $q^1 = u$ and $q^2 = v$ for $u = v = 0, w = 1$ and $w = -1$ we find respectively from (1.8)

$$xy (b - a) [z + \varepsilon (C - A - B)] = 0, \ xy (b - a) [z - \varepsilon (C - A - B)] = 0$$

whence it follows that the centre of mass of the body lies in the principal plane of the inertia ellipsoid constructed for the fixed point.

For example, let $x = 0$. It is convenient to take $q^1 = v, q^2 = w$ and the angle of precession as the generalized coordinates of the body. The condition (1.8) has the form of a polynomial in powers of v and w equal to zero

$$\begin{gathered}[yw - zv + \varepsilon (B - C) vw] [(a - c) y^2 + (a - b) z^2 + \\ (\varepsilon v)^2 (c - a) (B - A) (A + C - B) + (\varepsilon w)^2 (b - a) (C - A) (A + \\ B - C) + yv\varepsilon (c - a) (C + 2A - B) + zw\varepsilon (b - a) (B + 2A - C)] + \\ \varepsilon [y + \varepsilon (B - A) v] [z + \varepsilon (C - A) w] (b - c) (B + C - A) (1 - v^2 - w^2) = 0\end{gathered}$$

Therefore, all its coefficients should equal zero. A simple analysis of these identities shows that for $\varepsilon \neq 0$ it is possible only when the body is dynamically symmetric, and for $\varepsilon = 0$ if it is symmetric in two more cases: $x = y = z = 0$ (Euler's case), $B (C - A) y^2 = C (A - B) z^2$, $C < A < B$ or $C > A > B$ (Hess' case). Ony in these cases do manifolds of trajectories exist, locally specified by two equations of the form (1.5).

2. Consider the conditions under which Eq.(1.1) admits of the sole invariant relation $\langle \lambda_n (q), q^{\cdot} \rangle = 0$. Without loss of generality we assume that the vector $\lambda_n = (\lambda_{n|}{}^1, \ldots, \lambda_{n|}{}^n)$ is normalized, i.e. $\langle \lambda_n, \lambda_n \rangle = 1$.

Together with the vector field $\lambda_n (q)$ we can introduce a vector field $\lambda_1 (q), \ldots, \lambda_{n-1} (q)$ so that at each point $q \in M$ the vectors $\lambda_1, \ldots, \lambda_n$ form the orthogonal n-hedron $\langle \lambda_i, \lambda_j \rangle = \delta_{ij}$ $(i, j = 1, \ldots, n)$, where δ_{ij} is the Kronecker delta (see /2/).

Let us recall the notation used in /2/.

$$\gamma_{hkl} = \langle (\lambda_l, \nabla) \lambda_h, \lambda_k \rangle \quad (h, k, l = 1, \ldots, n)$$

The invariants γ_{hkl} are skew-symmetric with respect to the first two indices: $\gamma_{hkl} = -\gamma_{khl}$. The independent invariants among these define the components of the angular velocity of the n-hedron's rotation with its origin moving along the integral lines of the vector field $\lambda_1, \ldots, \lambda_n$ and are referred to as the Darboux-Ricci rotation coefficients.

From Eq.(1.7), taking into account (1.1), we obtain

$$\langle (q^{\cdot}, \nabla) \lambda_n, q^{\cdot} \rangle + \langle \lambda_n, F \rangle = \varkappa \langle \lambda_n, q^{\cdot} \rangle \tag{2.1}$$

hence it follows that the factor $\varkappa$ can be of the form $\langle \nu (q), q^{\cdot} \rangle$ only. Then the identity (2.1) leads to the equations

$$\langle \lambda_n, F \rangle = 0 \tag{2.2}$$

$$\nabla_s \lambda_{n/r} + \nabla_r \lambda_{n/s} = \nu_r \lambda_{n/s} + \nu_s \lambda_{n/r} \quad (r, s = 1, \ldots, n) \tag{2.3}$$

Multiplying (2.3) successively by $\lambda_{i|}{}^r \lambda_{j|}{}^s$ and performing a convolution with respect to the indices r and s, we obtain the equivalent equations in the invariant form

$$\gamma_{nij} + \gamma_{nji} = \delta_{jn} \sigma_i + \delta_{in} \sigma_j \quad (i, j = 1, \ldots, n) \tag{2.4}$$

where the unknowns ν_l are replaced by $\sigma_i = \langle \nu, \lambda_i \rangle$.

The analysis of these equations in general form is complicated. The case of $n = 2$ is discussed in /2/.

Let us consider the case when $n = 3$ under the condition that $F \neq 0$ in a certain domain $D \subseteq M$. Eqs.(2.2) and (2.4) yield

$$\gamma_{311} = \gamma_{322} = 0, \quad \gamma_{321} + \gamma_{312} = 0, \quad \langle \lambda_3, F \rangle = 0 \tag{2.5}$$

The unknowns $\sigma_1, \sigma_2, \sigma_3$ are determined from the remaining three equations of (2.4).

In accordance with the last equality in (2.5), we select $\lambda_1 = F/\langle F, F\rangle^{1/2}$. Then the vector of the curvature is $\tau = (\lambda_1, \nabla)\lambda_1 = \rho\lambda_2$. In fact, from (2.5) we have

$$\langle \lambda_3, \tau \rangle = \gamma_{311} + \langle \lambda_3, \tau \rangle = (\lambda_1, \nabla) \langle \lambda_3, \lambda_1 \rangle = 0$$

If $\rho \equiv 0$, the lines of force are geodesic and there exists an ensemble of two invariant relations of the form (1.5). This case was discussed in Sect.1. Let $\rho \neq 0$, then the vector fields λ_2 and λ_3 are mutually orthogonal, and this is exactly what is required.

Thus, at each point of the domain $D \subseteq M$, where the curvature of a line of force is not zero, we can introduce, as shown above, an orthogonal trihedron $\lambda_1, \lambda_2, \lambda_3 = \lambda_1 \times \lambda_2$. We shall call it a K-trihedron.

Theorem 2. In an open domain on the configuration manifold, at whose points a K-trihedron is defined, the sole invariant homogeneous linear relation of a holonomic system exists if and only if the coefficients of the trihedron's rotation satisfy the conditions

$$\gamma_{322} = 0, \quad \gamma_{312} + \gamma_{321} = 0 \tag{2.6}$$

In this case the invariant relation has the form

$$\det \| F, \eta, q^{\cdot} \| = 0 \quad (\eta = (F, \nabla) F)$$

Conditions (2.6) have a clear geometrical meaning. The first equation in (2.6) means that the vector of curvature of the integral lines in the field λ_2 is orthogonal to the vector λ_3. The second equation shows that the integral curves of the fields λ_1 and λ_2 form an ensemble which (in the sense of /2/) is canonical with respect to the integral curves of the field λ_3. It is well-known (see /4/) that both of these conditions are necessary for the integral curves of the field λ_3 to be the trajectories of the group of motions of Riemann's manifold $(M, \langle\ ,\rangle)$ of the system.

Let $n > 1$ be an arbitrary natural number, and the right-hand parts of the equation of motion (1.1) contain, apart from the position forces $F(q)$ the gyroscopic forces $\Omega q^{\cdot}$, where $\Omega(q)$ is an arbitrary two-order skew-symmetric tensor.

For the existence of an ensemble of invariant relations (1.5) when $m = n - 1$ and $m = 1$ it is necessary that

$$\Omega\lambda_p = 0, \quad p = n - m\delta_{n-1,m} \tag{2.7}$$

In fact, computing the total derivative of (1.5) with respect to time along the system's trajectory, we obtain

$$\langle (q^{\cdot}, \nabla)\lambda_\alpha, q^{\cdot}\rangle + \langle \lambda_\alpha, \Omega q^{\cdot} + F\rangle = \sum_{\beta=l+1}^{n} \varkappa_{\alpha\beta} \langle \lambda_\beta, q^{\cdot}\rangle \tag{2.8}$$
$$(l = n - m;\ \alpha = l + 1, \ldots, n)$$

Consequently, the multipliers $\varkappa_{\alpha\beta}$ can be of the form $\langle \nu_{\alpha\beta}(q), q^{\cdot}\rangle - \xi_{\alpha\beta}(q)$ only. Therefore, the identities (2.8) yield, in particular,

$$\Omega\lambda_\alpha = \sum_{\beta=l+1}^{n} \xi_{\alpha\beta}\lambda_\beta \quad (\alpha = l + 1, \ldots, n)$$

or, in the equivalent form,

$$\langle \Omega\lambda_\alpha, \lambda_i\rangle = \sum_{\beta=l+1}^{n} \xi_{\alpha\beta}\delta_{\beta i} \quad (\alpha = l + 1, \ldots, n;\ i = 1, \ldots, n)$$

(the vectors $\lambda_1, \ldots, \lambda_n$ constitute an orthogonal n-hedron). The conditions (2.7) follow from the above.

As a consequence we obtain that:

a) in systems with two degrees of freedom, for $\Omega \neq 0$ no invariant relations of the type discussed exist;

b) for $F \neq 0$, the assertion of Theorem 1 completed by the condition $\Omega F = 0$ remains valid;

c) since a definite pseudovector ω corresponds to an arbitrary skew-symmetric 3×3 matrix Ω (see /1/), in systems with three degrees of freedom, for $\Omega \neq 0$ the desired invariant relation can be of the form $\langle \omega, q^{\cdot}\rangle = 0$ only.

3. We shall now consider non-holonomic systems with the ideal linear constraints (1.5). Without loss of generality, we assume that the vectors $\lambda_{l+1}, \ldots, \lambda_n$ are unit vectors, and they are mutually orthogonal in the metrics which are defined by the system's kinetic energy. We will write the equations of motion in the form containing the multipliers μ_α of the constraints

$$g^{\cdot\cdot} = -(\Gamma g^{\cdot}, g^{\cdot}) + F + \sum_{\alpha=l+1}^{n} \mu_\alpha \lambda_\alpha \tag{3.1}$$

From Eqs.(1.5) and (3.1) we find

$$\mu_\alpha = -\langle (q^{\cdot}, \nabla)\, \lambda_\alpha, q^{\cdot}\rangle - \langle \lambda_\alpha, F\rangle \quad (\alpha = l+1, \ldots, n)$$

Substitution of the expressions obtained reduces the equations of motion to the form

$$g^{\cdot\cdot} = -(\Gamma g^{\cdot}, g^{\cdot}) - \sum_{\alpha=l+1}^{n} \lambda_\alpha \langle R_\alpha g^{\cdot}, g^{\cdot}\rangle + Q \tag{3.2}$$

$$R_\alpha = \| \nabla_s \lambda_{\alpha/r} \|, \quad Q = F - \sum_{\alpha=l+1}^{n} \lambda_\alpha \langle \lambda_\alpha, F\rangle$$

Each layer of the manifold $T_q M$ can be expanded in a direct sum of two linear vector subspaces, one of which $X_m(q)$, is extended over the vectors $\lambda_1, \ldots, \lambda_m$. The second subspace $Y_l(q)$ is of dimensions l, which equals the number of degrees of freedom of the system. We shall refer to it as a subspace of possible velocities for a specified point $q \in M$. The positional force $Q(q) \in \mathbf{R}^n$ on the right-hand side of (3.2) is a projection of the applied force $F(q)$ on the space $Y_l(q)$. The component of force F, equal to $F - Q$, is "quenched" by the reaction of the ideal constraints (1.3). The sum on the right in (3.2) is a reaction of the constraints when no outside active forces affect the system.

Notice that relations (1.5) are the first integrals of Eqs.(3.2) with zero integration constants. Let us consider the conditions under which there also exist for (3.2) k invariant relations of the form

$$\langle \lambda_{l-k+1}, q^{\cdot}\rangle = 0, \ldots, \langle \lambda_l, q^{\cdot}\rangle = 0 \ (1 \leqslant k \leqslant l-1) \tag{3.3}$$

where vectors $\lambda_{l-k+1}(q), \ldots, \lambda_l(q)$ are linearly independent at the points $q \in M$. Without loss of generality we can assume that the vectors $\lambda_{l-k+1}, \ldots, \lambda_l, \lambda_{l+1}, \ldots, \lambda_n$ are unit vectors and are mutually orthogonal. Still selecting appropriately $l-k$ regular vector fields $\lambda_1(q), \ldots, \lambda_{l-k}(q)$, we arrive at a family of orthogonal n-hedrons at the points $q \in M$.

In accordance with the definition in Sect.1, relations (1.5) and (3.3) constitute an invariant ensemble for the non-holonomic system discussed, if

$$\frac{d}{dt}\langle \lambda_\alpha, g^{\cdot}\rangle \equiv \sum_{\beta=l-k+1}^{n} \varkappa_{\alpha\beta}(g, g^{\cdot}) \langle \lambda_\beta, g^{\cdot}\rangle \quad (\alpha = l-k+1, \ldots, l) \tag{3.4}$$

where the generalized accelerations on the left-hand sides are replaced in accordance with (3.2).

It follows from the identities (3.4) that

$$\varkappa_{\alpha\beta} = \langle \nu_{\alpha\beta}(q), q^{\cdot}\rangle$$

$$\nabla_s \lambda_{\alpha/r} + \nabla_r \lambda_{\alpha/s} = \sum_{\beta=l-k+1}^{n} (\nu_{\alpha\beta/s}\lambda_{\beta/r} + \nu_{\alpha\beta/r}\lambda_{\beta/s}) \tag{3.5}$$

$$\langle \lambda_\alpha, Q\rangle = 0 \quad (\alpha = l-k+1, \ldots, l;\ r, s = 1, \ldots, n) \tag{3.6}$$

Multiplying (3.5) successively by $\lambda_{i|}{}^r \lambda_{j|}{}^s$ and performing a convolution with respect to the indices r and s, in invariant form we obtain the equations

$$\gamma_{\alpha ij} + \gamma_{\alpha ji} = \sum_{\beta=l-k+1}^{n} (\sigma_{\alpha\beta i}\delta_{\beta j} + \sigma_{\alpha\beta j}\delta_{\beta i}) \tag{3.7}$$

$$(\alpha = l-k+1, \ldots, l;\ \ i, j = 1, \ldots, n)$$

which are equivalent to Eqs.(3.5). In the above formula, the unknown vectors $\nu_{\alpha\beta}$ are replaced by the scalars $\sigma_{\alpha\beta i} = \langle \nu_{\alpha\beta}, \lambda_i\rangle$.

We will now consider the following special cases:

1) $k = l-1$, $l > 1$ is an arbitrary natural number;
2) $k = 1, l = 3$, and $Q \neq 0$ in a certain domain $D \subseteq M$.

In the first case Eqs.(3.7) yield

$$\gamma_{211} = \ldots = \gamma_{l11} = 0 \tag{3.8}$$

The remaining Eqs.(3.7) considered as algebraic equations with unknown $\sigma_{\alpha\beta i}$ are obviously independent.

In addition, it follows from conditions (3.6) that the vector fields λ_1 and Q are collinear.

Allowing for the geometrical sense of conditions (3.8), we formulate the following theorem.

Theorem 3. Let $Q \neq 0$ in a certain domain $D \subseteq M$. The non-holonomic system with $n - m > 1$ degrees of freedom, restricted by the ideal linear constraints, possesses in domain D an invariant ensemble of $n-1$ relations which are linear and homogeneous with repect to the velocities, when and only when at each point $q \in D$ the vector of curvature $\tau(q)$ of the integral curve of the field Q, which passes through this point, is orthogonal to the subspace of the possible velocities $Y_{n-m}(q)$. For $Q \equiv 0$, systems with two degrees of freedom always have such an invariant ensemble.

The last assertion of this theorem is obvious since its sole condition $\gamma_{311} = 0$ can be satisfied by an appropriate choice of the function $\varphi(q)$ in the expression $\lambda_1 = e_1(q)\cos\varphi + e_2(q)\sin\varphi$, where the vectors e_1 and e_2 form any orthogonal basis of the subspace $Y_2(q)$ at each point $q \in D$

Notice that when Eqs.(1.5) are fully integrable, Theorem 1 follows from Theorem 3. This is the result of the definition of a geodesic line contained in Riemann's manifold of the system (see /1/).

Consider the case where $k = 1,\ l = 3$. Eqs.(3.7) yield $\gamma_{311} = \gamma_{322} = 0,\ \gamma_{312} + \gamma_{321} = 0$. The remaining equations of (3.7) are used to determine the unknowns $\sigma_{\alpha\beta i}$.

Let us write $\lambda_1 = Q/\langle Q, Q\rangle^{1/2}$. The vector of curvature is

$$(\lambda_1, \nabla)\,\lambda_1 = x(q) + y(q) \quad (x \in X_{n-3},\ y \in Y_3)$$

When $y(q) \equiv 0$, a non-homonomic system has two relations of the form (3.3) which, together with the constraint equations, constitute the invariant ensemble (see Theorem 3).

Let $y(q) \neq 0$ in a certain domain $D_1 \subseteq D$. Then, connecting to the specified vector fields $\lambda_4, \ldots, \lambda_n$ of the constraint Eqs.(1.5) three more vector fields

$$\lambda_1,\ \lambda_2 = y/\langle y, y\rangle^{1/2},\quad \lambda_3 = (\varepsilon^{ij\ldots rs}\lambda_{4/j}\ldots\lambda_{n/r}\lambda_{1/s}\lambda_{2/p})$$

(ε is the n-valence fundamental Ricci and Levi-Civita tensor, see /5/), we obtain at each point $q \in D_1$ an orthogonal n-hedron which we shall call a key n-hedron. In this case the conditions (3.6) and $\gamma_{311} = 0$ are satisfied automatically:

$$\langle Q, \lambda_3\rangle = 0,\ \gamma_{311} = -\langle(\lambda_1, \nabla)\,\lambda_1, \lambda_3\rangle = -\langle x + y, \lambda_3\rangle = 0$$

Theorem 4. In an open domain in the n-dimensional configuration manifold at whose points the key orthogonal n-hedron is defined, the non-holonomic system with three degrees of freedom and with constraints (1.5) possesses an invariant ensemble of $(n-2)$ relations that are uniform and linear with respect to the velocities, if and only if the rotation coefficients of the n-hedron satisfy the conditions

$$\gamma_{322} = 0,\quad \gamma_{312} + \gamma_{321} = 0 \tag{3.9}$$

This ensemble is formed by the inequalities (1.5) and the relation

$$\det\|Q, \eta, \lambda_4, \ldots, \lambda_n, q^{\cdot}\| = 0 \quad (\eta = (Q, \nabla)\,Q)$$

4. As mentioned above, each particular solution of the differential Eqs.(3.2) lies on the submanifold given by the equations $\langle\lambda_{l+1}, q^{\cdot}\rangle = c_{l+1}, \ldots, \langle\lambda_n, q^{\cdot}\rangle = c_n$, where $c_{l+1}, \ldots, c_n$ are constants, and $l = n - m$. However, only those solutions which occupy the submanifold (1.5) correspond to the actual motion of the system. Therefore, in determining the whole multiplicity of the motions of a mechanical system with the constraints (1.5) we must search not for the first integrals of Eqs.(3.2) but for the relations $\varphi(q, q^{\cdot}) = \text{const}$, for which the derivative

$$\frac{d\varphi}{dt} = \left(\frac{\partial\varphi}{\partial q}, q^{\cdot}\right) + \left(\frac{\partial\varphi}{\partial q^{\cdot}}, -(\Gamma q^{\cdot}, q^{\cdot}) - \sum_{\alpha=l+1}^{n} \lambda_\alpha \langle R_\alpha q^{\cdot}, q^{\cdot}\rangle + Q\right)$$

has the form

$$\frac{d\varphi}{dt} = \sum_{\alpha=l+1}^{n} \varkappa_\alpha(q, q^{\cdot})\,\langle\lambda_\alpha, q^{\cdot}\rangle \tag{4.1}$$

($\varkappa_\alpha$ are any continuous functions).

It may happen that the relation

$$\rho(q)\,\langle\lambda_l, q^{\cdot}\rangle = \text{const} \quad (\rho \neq 0)$$

of the above-mentioned nature, corresponds to the relation $\langle\lambda_l, q^{\cdot}\rangle = 0$ which together with Eq.(1.5) forms an invariant ensemble (see the definition in Sect.1) for the equations of motion (3.2). Then, on substituting $\varphi = \rho\langle\lambda_l, q^{\cdot}\rangle$, into (4.1), we find as in Sect.3 that

$$\varkappa_\beta = \langle\nu_\beta(q), q^{\cdot}\rangle \quad (\beta = l + 1, \ldots, n)$$

$$\langle \mathbf{grad}\,\rho, \lambda_i\rangle\,\delta_{lj} + \langle\mathbf{grad}\,\rho, \lambda_j\rangle\,\delta_{li} + \rho(\gamma_{lij} + \gamma_{lji}) = \sum_{\beta=l+1}^{n} (\sigma_{\beta i}\delta_{\beta j} + \sigma_{\beta j}\delta_{\beta i}) \quad (i, j = 1, \ldots, n) \tag{4.2}$$

$$\langle\lambda_l, Q\rangle = 0 \tag{4.3}$$

For $i, j = 1, \ldots, l - 1$, Eqs.(4.2) and (4.3) are satisfied automatically since (see above) the relation $\langle\lambda_l, q^{\cdot}\rangle = 0$ forms with (1.5) an invariant ensemble. For $i = 1, \ldots, n;\ j = l + 1, \ldots, n$ the inequalities (4.2) can serve to find the unknown scalars $\sigma_{\beta i} = \langle\nu_\beta, \lambda_i\rangle$. Then there remain the equations $(\theta = \ln\rho^{-1})$

$$\langle \operatorname{grad} \theta, \lambda_\alpha \rangle = \gamma_{l\alpha l} \quad (\alpha = 1, \ldots, l) \tag{4.4}$$

which for $i = 1, \ldots, l; j = l$ follow from (4.2).

Eqs.(4.4) are of the first order in the unknown function $\theta(q)$. The compatibility of such equations can be investigated by the usual methods (/6/).

Thus, the following assertion is proved: for the existence in domain $D \subseteq M$ of a non-zero vector field $\xi(q) \in Y_l(q)$ such that any particular solution $q(t) \in D$ of Eqs.(3.2), which corresponds to the actual motion of the system, satisfies in domain D the relation

$$\langle \xi, q^{\cdot} \rangle = \langle \xi, q^{\cdot} \rangle |_{t=t_0}$$

it is necessary and sufficient that the constraint Eqs.(1.5) and $\langle \lambda_l, q^{\cdot} \rangle = 0 \quad (\lambda_l = \xi / \langle \xi, \xi \rangle^{1/2})$ for (3.2) constitute an invariant compatibility in the sense of the definition in Sect.1, and the system of Eqs.(4.4) is compatible with respect to the function $\theta = -{}^1/_2 \ln \langle \xi, \xi \rangle$.

We note that on the right in Eqs.(4.4) we have the components of the projection of the vector of curvature $\tau_l = \gamma_{lil} \lambda_i$ of the integral line of the field λ_l in the subspace of possible velocities.

In the case of holonomic systems with n degrees of freedom, the first integral (general) $\rho(q) \langle \lambda_n, q^{\cdot} \rangle = \text{const}$ of the equations of motion (1.1) may correspond to the sole invariant relation $\langle \lambda_n, q^{\cdot} \rangle = 0$ Arguing as in Sect.2 we can show that in this case the unknown function $\rho(q)$ is determined from the system of equations

$$\langle \operatorname{grad} \rho, \lambda_i \rangle + \rho \gamma_{nin} = 0 \quad (i = 1, \ldots, n)$$

which means that the vector of curvature of the integral line of the field λ_n should be a gradient.

Example 2. A solid rotates about a fixed point in the uniform gravitational field

$$U = xu + yv + zw \quad (x^2 + y^2 + z^2 \neq 0)$$

the body motion being subject to the ideal non-holononic constraint

$$pe_1 + qe_2 + re_3 = 0 \tag{4.5}$$

where (p, q, r) are the components of the instantaneous angular velocity ω of the body along the principal axes of inertia constructed for the fixed point, and (e_1, e_2, e_3) are the components along these axes of a certain direction e which is fixed in the body,

$$(e_1)^2 + (e_2)^2 + (e_3)^2 = 1 \tag{4.6}$$

For the other notation see, for example, /1/. The body weight is assumed to be equal to unity. Two different methods for realizing constrain (4.5) are given in /7, 8/.

The system under consideration has two degrees of freedom. Let us clarify whether a relation of the following form exists:

$$\alpha_1 p + \alpha_2 q + \alpha_3 r = 0 \tag{4.7}$$

which would, teogether with (4.5), constitute an invariant ensemble. Here α_i are any functions of the generalized coordinates of the system, e.g. of the Euler angles.

It is convenient to conduct further calculation in non-holonomic local coordinates $d\beta^1 = pdt, d\beta^2 = qdt, d\beta^3 = rdt$. The metrics generated by the kinetic energy of the system (ignoring the non-holonomic constraint) is

$$ds^2 = A\,(d\beta^1)^2 + B\,(d\beta^2)^2 + C\,(d\beta^3)^2 \tag{4.8}$$

Let us agree to denote the scalar product in this metric by $\langle , \rangle$ and in the Euclidean metric by $(,)$.

The covector of the active force F has the components

$$\frac{\partial U}{\partial \beta^1} = yw - zv, \quad \frac{\partial U}{\partial \beta^2} = zu - xw, \quad \frac{\partial U}{\partial \beta^3} = xv - yu$$

We denote by $Q = F - \langle F, e \rangle / E$, where $E^2 = \langle e, e \rangle = a\,(e_1)^2 + b\,(e_2)^2 + c\,(e_3)^2$, the projection of the force F on the subspace of possible velocities at each point of the configuration manifold $SO(3)$ of the system.

Consider the vector $\iota = (Q, \nabla) Q$ equal to $\langle Q, Q \rangle \tau + \pi \lambda_1$, where τ is the curvature vector of the field line $Q, \lambda_1 = Q / \langle Q, Q \rangle^{1/2}$, and π is a certain scalar function. Since $\langle \tau, Q \rangle = \langle \lambda_1, F \times e \rangle = 0$, in conformity with Theorem 3, for an invariant ensemble (4.5), (4.7) to exist it is necessary and sufficient that the sole condition

$$\langle \iota, F \times e \rangle = 0 \tag{4.9}$$

be satisfied.

To find the covariant components of the vector ι generated by the field Q, while omitting the cumbersome calculations of the constraint coefficients of the Riemann manifold, it is worth noting that the left-hand sides of the dynamic Euler equations represent the components of a similar covector generated by the angular velocity field, $(p, q, r) = (\omega^1, \omega^2, \omega^3) = (\beta^{\cdot 1}, \beta^{\cdot 2}, \beta^{\cdot 3})$. For example, /9/

$$A\omega^{\cdot 1} + (C - B)\,\omega^2\omega^3 = A \frac{\partial \omega^1}{\partial \beta^i}\,\omega^i + (C - B)\,\omega^2\omega^3 = (\omega, \nabla)\,\omega_1$$

Direct calculations show that condition (4.9) has the form of a homogeneous third-degree polynomial equal to zero, with respect to the direction cosines u v and w of the vertical, which are connected by the single relation

$$u^2 + v^2 + w^2 = 1 \tag{4.10}$$

Consequently, allowing for all possible absolute values and the combinations of the signs of u, v and w, which satisfy the identity (4.10), all the coefficients of this manifold should be zero. This condition adds to (4.6) ten more equations in the parameters $x, y, z, A, B, C, e_1, e_2, e_3$ of the system

$$G\,[z^2be_2ce_3 + yz\,(c\zeta - b\eta) - y^2be_2ce_3] + J\,(y^2c\zeta + 2yzbe_2ce_3 + z^2b\eta) = 0 \tag{4.11}$$

$$G\,[(y^2 - z^2)\,ae_1ce_3 + 2ybe_2\,(xce_3 - zae_1) + xz\,(b\eta - c\zeta)] + 2J\,(z^2ae_1be_2 - xzbe_2ce_3 - yzae_1ce_3 - xyc\zeta) + I\,(y^2c\zeta + 2yzbe_2ce_3 + z^2b\eta) = 0 \tag{4.12}$$

$$G\,[(y^2 - z^2)\,ae_1be_2 + 2zce_3\,(yae_1 - xbe_2) + xy\,(b\eta - c\zeta)] + 2J\,(y^2ae_1ce_3 - yzae_1be_2 - xybe_2ce_3 - xzb\eta) + N\,(y^2c\zeta + 2yzbe_2ce_3 + z^2b\eta) = 0 \tag{4.13}$$

$$G\,[(y^2 - z^2)\,a\xi + (z^2 - x^2)\,b\eta + (x^2 - y^2)\,c\zeta] + 2J\,(x^2be_2ce_3 - xyae_1ce_3 - xzae_1be_2 - yza\xi) + 2I\,(y^2ae_1ce_3 - yzae_1be_2 - xybe_2ce_3 - xzb\eta) + 2N\,(z^2ae_1be_2 - xzbe_2ce_3 - yzae_1ce_3 - xyc\zeta) = 0 \tag{4.14}$$

Here

$$G = xe_1 + ye_2 + ze_3, \quad J = ye_3\left(1 - \frac{c}{E^2}\right) - ze_2\left(1 - \frac{b}{E^2}\right)$$

$$I = ze_1\left(1 - \frac{a}{E^2}\right) - xe_3\left(1 - \frac{c}{E^2}\right), \quad N = xe_2\left(1 - \frac{b}{E^2}\right) - ye_1\left(1 - \frac{a}{E^2}\right)$$

$$\xi = E^2 - a\,(e_1)^2, \quad \eta = E^2 - b\,(e_2)^2, \quad \zeta = E^2 - c\,(e_3)^2$$

The remaining six equations are obtained from (4.11), (4.12) and (4.13) by the cyclic rearrangement $a \to b \to c \to a,\ x \to y \to z \to x,\ e_1 \to e_2 \to e_3 \to e_1$. As a consequence we have $G \to G,\ J \to I \to N \to J,\ \xi \to \eta \to \zeta \to \xi$.

The equations obtained connect the system's parameters and admit of the solution

$$\frac{x}{e_1\,(E^2 - a)} = \frac{b}{e_2\,(E^2 - b)} = \frac{z}{e_3\,(E^2 - c)} \tag{4.15}$$

Obviously, these formulae lose their meaning only in the case where the covector e is directed along one of the principal axes of the body's inertia, constructed for the fixed point.

Let us examine this case. Suppose, for example, that $e_1 = e_2 = 0, e_3 = 1$. Then $G = z, J = I = N = 0$. Eq.(4.11) yields $yz^2bc = 0$.

If $z \neq 0$, then taking into account the Eq.(4.12) $xz^2bc = 0$ it is necessary that $x = y = 0$. Then from the equation (4.14) $z^3\,(b - a)\,c = 0$ we obtain that $A = B$. This is the Lagrange case. The constraint (4.5) does not exert any influence on the body: it only limits the initial conditions of its motions, /10/.

If $z = 0$, we have one more general case of the integrability of the non-holonomic system in question, found by Kharlamova, /11/. In this case the equations of motion admit of the first integral of $Apx + Bqy = \text{const}$ (for ecample, the equations in the Voronets form).

We will further assume that the covector e is not collinear with any of the principal directions of the energy tensor which refers to the fixed point of the body. Thus, if the body centre of mass is situated on an axis with the direction vector

$$\lambda_2 = [e_1\,(E^2 - a)/m_2,\ e_2\,(E^2 - b)/m_2,\ e_3\,(E^2 - c)/m_2]$$
$$((m_2)^2 = A\,[e_1\,(E^2 - a)]^2 + B\,[e_2\,(E^2 - b)]^2 + C\,[e_3\,(E^2 - c)]^2)$$

then a relation of the form (4.7) exists, which together with the constraint Eq.(4.5) forms an invariant ensemble for the equations of motion with a multiplier. Here the vectors $\lambda_3 = e$ and $\lambda_1 = Q/\langle Q, Q\rangle^{1/2}$ have the components

$$\lambda_3 = (ae_1/E,\ be_2/E,\ ce_3/E)$$
$$\lambda_1 = [e_2e_3\,(B - C)/m_1,\ e_3e_1\,(C - A)/m_1,\ e_1e_2\,(A - B)/m_1]$$
$$((m_1)^2 = A\,[e_2e_3\,(B - C)]^2 + B\,[e_3e_1\,(C - A)]^2 + C\,[e_1e_2\,(A - B)]^2)$$

In conformity with what we said in Sect.3, the coefficients of relation (4.7) should be proportional to the components of the covector which is orthogonal to the vectors e and F. Since $n = 3$, relation (4.7) is fully defined by this condition, and can be written as

$$Ae_1\,(E^2 - a)p + Be_2\,(E^2 - b)\,q + Ce_3\,(E^2 - c)\,r = 0 \tag{4.16}$$

or, using (4.5),

$$Ape_1 + Bqe_2 + Cre_3 = 0 \tag{4.17}$$

Eqs.(4.5) and (4.17) mean that

$$p/\lambda_{1|}{}^1 = q/\lambda_{1|}{}^2 = r/\lambda_{1|}{}^3 \tag{4.18}$$

i.e. in the corresponding motion of the system the diections of the angular velocity and kinetic moment are fixed in the body.

Let us find whether the function $\rho(u, v, w) \neq 0$ can be selected so that

$$d/dt\,[\rho\,(Ape_1 + Bqe_2 + Cre_3)] = (pe_1 + qe_2 + re_3)\,\varkappa$$

In our case, Eqs.(4.4) have the form

$$(X_1\theta + \gamma_{122} =) \frac{\partial\theta}{\partial\beta^i}\lambda_1|^i + \gamma_{122} = 0 \tag{4.19}$$

$$(X_2\theta =) \frac{\partial\theta}{\partial\beta^i}\lambda_2|^i = 0 \tag{4.20}$$

where

$$\gamma_{122} = [(A - B)^2 (E^2 - a)(E^2 - b)(e_1e_2)^2 + (B - C)^2(E^2 - b)(E^2 - c)\cdot (e_2e_3)^2 + (C - A)^2(E^2 - c)(E^2 - a)(e_3e_1)^2]/[m_1\,(m_2)^2]$$

With this assumption regarding the direction of the vector e, the constant $\gamma_{122} \neq 0$ is different from zero.

Constructing the Poisson bracket $(X_1X_2 - X_2X_1)\,\theta = 0$, we obtain

$$(X_3\theta =) \frac{\partial\theta}{\partial\beta^i}\lambda_3|^i = 0$$

Constructing the second bracket $(X_2X_3 - X_3X_2)\,\theta = 0$, we find that $X_1\theta = 0$. This proves the inconsistency of the system of Eqs.(4.19) and (4.20).

It is interesting that in the given problem with a non-holonomic constraint in all three cases (i.e. of Lagrange, of that discussed in /11/, and of that considered above) the additional integral is a surface integral, and at the same time in the second and third cases it is a generalized surface integral (see /12/), but in our case it is the particular integral (4.17).

We shall prove that in other cases the invariant ensemble (4.5), (4.7) does not exist. In accordance with the argument above, this assertion remains to be proved when covector e does not coincide with the principal axis of inertia at the fixed point. Let us reformulate Theorem 3 as follows: for the existence of the invariant ensemble it is necessary and sufficient that the integral curves of the vector field Q are the solution of Eq.(3.2) when $Q = 0$.

In the absence of outside active forces, the body, restricted by the non-holonomic constraint (4.5), rotates with angular velocity /7, 8/

$$\Omega = -\lambda_1 v\,\mathrm{th}\,v\,(\varkappa t + k) - \lambda_2\frac{v}{\mathrm{ch}\,v\,(\varkappa t + k)}$$

$$\varkappa^2 = \frac{E^2\,[A\,(e_1)^2 + B\,(e_2)^2 + C\,(e_3)^2] - 1}{E^4ABC}, \quad k = \frac{\mathrm{arth}\cos\alpha}{v}$$

(the constants v and α are the initial velocity and the angle between the intial direction of the velocity and the vector $-\lambda_1$ respectively).

The vectors Q and $\Omega\,(t)$ should be collinear at each instant of time; Q is the linear vector-function of the variables $u\,(t), v\,(t)$ and $w\,(t)$ which are the solution of the Poisson equations $n^{\cdot} + \Omega \times n = 0,\ n = (u, v, w)$. Since at a fixed instant of time the vector n can have an arbitrary direction in the body, the collinearity condition yields

$$\frac{x}{\Omega_2e_3 - \Omega_3e_2} = \frac{y}{\Omega_3e_1 - \Omega_1e_3} = \frac{z}{\Omega_1e_2 - \Omega_2e_1}$$

Clearly, when $\alpha \neq 0$ the relations obtained are contradictory. For $\alpha = 0$ they are identical with (4.15). This proves the assertion.

In /13/, the generalization of the problem for a heavy gyrostat was discussed. Assuming that there exists an invariant relation of the form $\omega_1 = \omega_1^0\,(=\mathrm{const})$, where ω_1 is the projection of the angular velocity on the direction $O\xi_1$ fixed in the body, a Cartesian coordinate system $O\xi_1\xi_2\xi_3$ was introduced in /13/ in which the non-holonomic constraint (4.5) is described by the equation $\omega_3 = 0$, and the following conditions were obtained for the existence of the invariant relation above:

$$A_{31}\omega^0_1 + L = 0 \tag{4.21}$$

$$A_{32} = 0, \quad l_3 = 0, \quad A_{12}l_1 + A_{22}l_2 = 0 \tag{4.22}$$

Here, l_j are the components of the body centre of mass along the axes $O\xi_1\xi_2\xi_3$, the constant L is the projection of the gyrostatic moment on the $O\xi_3$ axis and A_{rs} are the corresponding moments of inertia of the body for the fixed point.

If $A_{31} = 0$, we have an integrable case, /11/. Let $A_{31} \neq 0$. Obviously, the conditions above do not impose any limitations on the body's inertia tensor and the three conditions (4.22) are equivalent to two relations (4.15). In fact, the first condition of (4.22) unambiguously defines the direction λ_1 of the $O\xi_2$ axis, but then the remaining two conditions are identical with (4.15), with $A_{31} = m_1\,(\lambda_1, \lambda_1)^{1/2}$.

For a body with centre of mass located on the axis (4.15), the equations of motion are integrated under the initial conditions $\omega_1^0 = -L/A_{31}$, in elliptic time functions. When $L = 0$, the corresponding motions of the body belong to the class of precessions of general form.

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ON THE STABILITY OF STATIONARY MOTIONS OF NON-CONSERVATIVE MECHANICAL SYSTEMS*

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The problem of the stability of stationary motions (SM) of mechanical systems admitting of first integrals and a function that does not grow along the motions is considered. Theorems are proposed on the stability and asymptotic stability in parts of the variables, as well as on the instability of the SM of such systems. The general situations are illustrated with an example of the motion of a heavy inhomogeneous sphere over a plane with friction.

1. We consider a scleronomic mechanical system that admits of time-independent first integrals $U_1(x) = c_1, \ldots, U_k(x) = c_k$, and a time-independent function $U_0(x)$ that does not grow along the system motions.

We assume that the functions $U_0(x), U_1(x), \ldots, U_k(x)$ are continuously differentiable with respect to the variables $x = (x_1, \ldots, x_n)$ therein. All or certain generalized coordinates and velocities or momenta of the system, quasicoordinates, certain functions of these quantities etc., can be these variables.

Theorem 1. If a function $U_0(x)$ that does not grow along the system motions has a strict local minimum for constant values of the integrals $U_i(x) = c_i$ $(i = 1, \ldots, k)$ of this system, then the values of the variables making this function a minimum correspond to the stable real motion of the system (this motion is usually called stationary).

Theorem 2. If the stationary motion (SM) makes the function $U_0(x)$ a strict local minimum and is isolated for constant values of the integrals $U_i(x) = c_i (i = 1, \ldots, k)$ of the motions along which the function $U_0(x)$ remains constant, then every perturbed motion that is sufficiently close to the unperturbed motion will tend asymptotically as $t \to \infty$ to one of the system SM, the corresponding strict local minimum of the function $U_0(x)$ for perturbed

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